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A Series Expansion for the First Positive Zero of the Bessel Functions

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Abstract. It is shown that the first positive zero $j_{\nu,1}$ of the Bessel function $J_\nu(x)$ is given by

$$j_{\nu,1} = 2(\nu+1)^{1/2}\left[1 + \frac{(\nu+1)}{4} - \frac{7(\nu+1)^2}{96} + \frac{49(\nu+1)^3}{1152} - \frac{8363(\nu+1)^4}{276480} + \cdots\right]$$

for $-1 < \nu < 0$.

1. It is well known that, when ν is real and $\nu > -1$, the Bessel function $J_\nu(x)$ has an infinite number of zeros and that all zeros are real (Watson [9]). We denote the sth positive zero of $J_\nu(x)$ by $j_{\nu,s}$.

Several approximations, asymptotic expansions or bounds for the zeros of Bessel functions exist (see [1], [2], [4], [6], [7], [9]). Especially McMahon's expansion for large zeros (see Abramowitz and Stegun [1]), Olver's asymptotic expansion for large orders and Olver's uniform asymptotic expansions (see Olver [6]) are interesting formulas, but, unfortunately, they are not applicable when s and ν are small. The purpose of this note is to give a series expansion for $j_{\nu,1}$ when $-1 < \nu < 0$.

2. Cayley [3] noticed that Graeffe's method for solving a polynomial equation can be applied for the efficient computation of

$$\sum_{s=1}^{\infty} j_{\nu,s}^{-2r} \equiv \sigma_\nu^{(r)}, \qquad r = 1,2,\ldots. \tag{1}$$

An upper bound for $j_{\nu,1}$ is given by Chambers [4]:

$$j_{\nu,1} < (\nu+1)^{1/2}\left[(\nu+2)^{1/2} + 1\right]. \tag{2}$$

Further it is known that, when $k > 1$,

$$\lim_{\nu \to -1} j_{\nu,k} = j_{1,k-1} > 0. \tag{3}$$

Thus, the first term in the left side of (1) is dominant when $\nu \simeq -1$, so that

$$j_{\nu,1} = 2(\nu+1)^{1/2}\phi_r(\nu) + o\left((\nu+1)^{r-1}\right), \qquad \nu \to -1, \tag{4}$$

where

$$\phi_r(\nu) = \left[\frac{1}{2^{2r}(\nu+1)^r \sigma_\nu^{(r)}}\right]^{1/2r} \tag{5}$$

is analytic at $\nu = -1$.

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By approximating $\phi_r(\nu)$ by a Taylor polynomial, we obtain

$$j_{\nu,1} = 2(\nu+1)^{1/2} \sum_{k=0}^{r-1} C_k(\nu+1)^k + o\big((\nu+1)^{r-1}\big), \qquad \nu \to -1, \tag{6}$$

where

$$C_k = \frac{1}{k!} \left. \frac{d^k \phi_r(\nu)}{d\nu^k} \right|_{\nu=-1} \tag{7}$$

is independent of r.

When $r \to \infty$, (6) becomes a series expansion for $j_{\nu,1}$, which, because of the presence of a branchpoint of $\phi_r(\nu)$ at $\nu = -2$, converges only in the interval $-1 < \nu < 0$.

Using REDUCE, which is a computer language for formula manipulation [5], we have computed C_k, $k = 0, 1, 2, 3, 4$, using (7), where $r = 8$ and

$$\phi_8(\nu) = \left[\frac{(\nu+2)^4(\nu+3)^2(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)}{429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738} \right]^{1/16}. \tag{8}$$

The result is

(9)

$$j_{\nu,1} = 2(\nu+1)^{1/2}\left[1 + \frac{(\nu+1)}{4} - \frac{7(\nu+1)^2}{96} + \frac{49(\nu+1)^3}{1152} - \frac{8363(\nu+1)^4}{276480} + \cdots\right].$$

In Table 1, we compare the exact values of $j_{\nu,1}$ with the approximate values given by (9), for $\nu = -3/4, -2/3, -1/2, -1/3, -1/4$ and also for $\nu = 0$ (although we were not able to prove the convergences of the expansion for $\nu = 0$).

TABLE 1

Values of the first zero $j_{\nu,1}$ of $J_\nu(x)$

ν	exact	approximation (9)
$-3/4$	1.058508	1.058489
$-2/3$	1.243046	1.242958
$-1/2$	1.570796	1.570056
$-1/3$	1.866351	1.863061
$-1/4$	2.006300	2.000273
0	2.404826	2.378740

3. An interesting application of (9) is the estimation of the smallest zero of Laguerre- and Gegenbauer-polynomials [8]. For example, the smallest zero ξ_n of the Laguerre polynomials $L_n^{(\alpha)}(x)$ is approximated by (see Tricomi [8])

$$\xi_n \simeq \frac{j_{\alpha,1}}{4k_n}\left[1 + \frac{2(\alpha^2-1) + j_{\alpha,1}^2}{48k_n^2}\right], \tag{10}$$

where $k_n = n + (\alpha+1)/2$. In Table 2, this approximation, where $j_{\alpha,1}$ is replaced by (9), is compared with the exact value of ξ_n.

TABLE 2

Values of the smallest zero of $L_n^{(\alpha)}(x)$

α	n	exact	approximation (10)
$-3/4$	3	0.089682	0.089679
	15	0.018520	0.018519
$-1/2$	3	0.190163	0.189982
	15	0.040452	0.040415
$-1/4$	3	0.299347	0.297530
	15	0.065463	0.065071
0	3	0.415775	0.406686
	15	0.093308	0.091294

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